Geometry of Pinchuk's map

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Abstract

Sergey Pinchuk found a polynomial map from the real plane to itself which is a local diffeomorphism but is not one-to-one. The aim of this paper is to give a geometric description of Pinchuk's map.

1 Introduction

In the paper [7] Pinchuk gave an example of a polynomial map $F : \mathbb{R}^2 \to \mathbb{R}^2$ with a non-vanishing Jacobian such that F is not a global diffeomorphism. Let us recall his construction. We define the following auxiliary polynomials in variables x, y:

$$t = xy - 1, \quad h = t(xt+1), \quad f = (xt+1)^2(t^2+y) \tag{1}$$

and a polynomial

$$u(f,h) = (1/4)f(75f^3 + 300f^2h + 450fh^2 + 276f^2 + 828fh + 48h^2 + 364f + 48h).$$

Then $F = (p, q)$ where p and q are given by

$$p = f + h, \tag{2}$$
$$q = -t^2 - 6th(h+1) + u(f,h). \tag{3}$$

Our aim is to give a geometric description of Pinchuk's map. As in [2] we find the set of points at which F is not proper. It allows us to divide the image of F into sets with constant multiplicity of fibers. Finally we illustrate how F transforms the real plane to these sets.

2 Geometry of Pinchuk's map

Let $f : X \to Y$ be a continuous map of locally compact spaces. We say that the mapping f is not proper at a point $y \in Y$, if there is no a neighborhood U of a point y such that the set $f^{-1}(cl(U))$ is compact.

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The set S_f of points at which the map f is not proper indicates how the map f differs from a proper map. In particular the restriction of f from $X \setminus f^{-1}(S_f)$ to $Y \setminus S_f$ is proper. This notion was studied in [5] in the complex case and in [6], where S_f is called an asymptotic variety.

Let $C = \Phi(\mathbb{R})$ be a curve given by

$$\Phi : \mathbb{R} \ni s \to (s^2 + 2s, u(s^2 + s, s)) \in \mathbb{R}^2$$

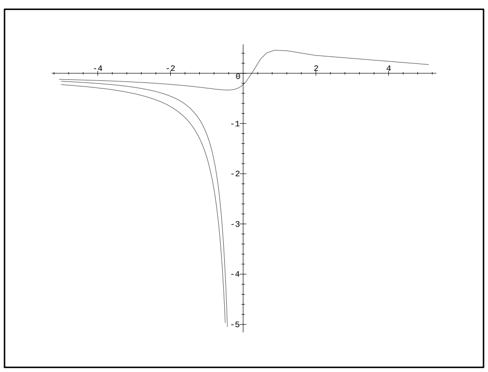
Theorem 2.1 *The set of points at which F is not proper is the curve C. The set $F^{-1}(C)$ is a smooth curve having three connected components. Furthermore*

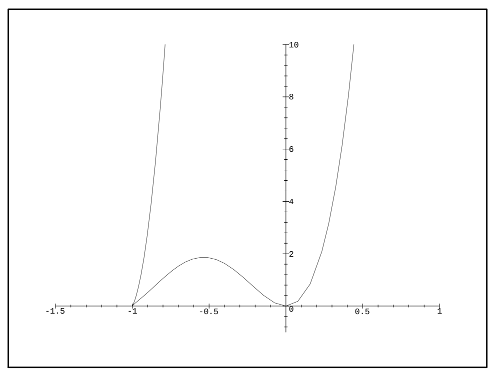
(i) for every $v \in \mathbb{R}^2 \setminus C \quad \#F^{-1}(v) = 2$,

(ii) for every $v \in C \setminus \{(-1, 0), (0, 0)\} \quad \#F^{-1}(v) = 1$,

(iii) $F^{-1}(-1, 0) = F^{-1}(0, 0) = \emptyset$.

A behavior of F is illustrated in figure below. A set $\mathbb{R}^2 \setminus F^{-1}(C)$ has 4 connected components. The marked ones are mapped diffeomorphically to a marked area above the curve C. Two unmarked regions are mapped diffeomorphically to an area below C. Indicated connected components of a curve $F^{-1}(C)$ are mapped to indicated parts of the curve C, respectively.





A similar description of Pinchuk's map was given in [2]. The author characterizes there another mapping from the class discovered by Pinchuk.

Note that C is not an algebraic curve although it has a polynomial parameterization. It follows from the proof of Lemma 3.6 that the Zariski closure of C in $\mathbb{R}^2$ consists of C and a point $\Phi(s)$, where s is a complex solution of an equation $75s^2 + 150s + 104 = 0$.

This shows that Theorem 3 in [6], where it is claimed that the asymptotic variety is algebraic, is partially false. A polynomial map $(x, y) \to (x^2, xy)$ is another counterexample. An asymptotic variety of this map is a right half line $[0, \infty) \times \{0\}$. Moreover points (b) and (c) in Theorem 3 can be removed. We refer the interested reader to [3], Theorem 3.3.

Our approach is based on the following observation due to Zbigniew Jelonek.

Proposition 2.2 *Let $\mathbb{C}F : \mathbb{C}^2 \to \mathbb{C}^2$ be a complexification of F. Then the set S_F of points at which F is not proper is contained in the set $S_{\mathbb{C}F}$ of points at which the mapping $\mathbb{C}F$ is not proper.*

Proof. Easy exercise. ■

The next result concerning complex polynomial maps was proved in [4], [5]:

Theorem 2.3 *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a dominant polynomial map. Then the set S_f of points at which f is not proper consists of a finite number (possibly 0) of affine algebraic curves.*

To prove Theorem 2.1 we find the set of points at which complex Pinchuk's map is not proper and then apply Proposition 2.2.

Proposition 2.4 $S_{\mathbb{C}F} = \Phi(\mathbb{C})$.

3 Proofs

Lemma 3.1 *The following relations hold for t, h, f, q:*

$$(h-t)f = h^2(h+1) \tag{4}$$

$$Q(f,h,q) = 0 \tag{5}$$

where $Q(f,h,q) = f^2(q - u(f,h)) + h^2(f - h(h+1))(f + (6f-h)(h+1))$.

Proof. The formula (4) follows immediately from (1). Multiplying equation (3) by f^2 gives $f^2(q-u(f,h)) + (tf)^2 + 6(tf)h(h+1)f = 0$. By (4) we have $tf = h(f - h(h+1))$. Substituting the right–hand side into above formula we obtain (5). ■

Consider a system of equations

$$\begin{cases} \bar{f} + \bar{h} = a \\ Q(\bar{f}, \bar{h}, b) = 0 \end{cases} \tag{6}$$

To solve this system for a given (a,b) it suffices to find roots of $W(\bar{f}, a, b)$ where $W(\bar{f}, a, b) = Q(\bar{f}, a - \bar{f}, b)$ is a monic polynomial of degree 6 with respect to $\bar{f}$. Therefore a number of solutions of (6) equals a number of roots of a polynomial $W(\bar{f}, a, b)$.

For every $(\bar{f}, \bar{h})$ with $\bar{f} \neq 0$ there is a unique pair (a,b) such that $(\bar{f}, \bar{h}, a, b)$ satisfies (6). Namely

$$\begin{cases} a = \bar{f} + \bar{h} \\ b = u(\bar{f}, \bar{h}) - \bar{h}^2(\bar{f} - \bar{h}(\bar{h}+1))(\bar{f} + (6\bar{f} - \bar{h})(\bar{h}+1))/\bar{f}^2 \end{cases} \tag{7}$$

Observe that, generically, the Pinchuk mapping is a composition of a polynomial map (f,h) and a map G given by (7). We precise this idea in Lemmas 3.3 and 3.4. First we need some preparation.

Let $B = \{\, (\bar f, \bar h) \in \mathbb{C}^2 : \bar f(\bar f - \bar h(\bar h + 1)) = 0 \,\}$.

Lemma 3.2 *The restriction of $(f,h) : \mathbb{C}^2 \to \mathbb{C}^2$ from $\mathbb{C}^2 \setminus f^{-1}(0)$ to $\mathbb{C}^2 \setminus B$ is a homeomorphism. A curve $f^{-1}(0)$ has two algebraic components: $A_1 = (f,h)^{-1}(0,0)$ and $A_2 = (f,h)^{-1}(0,-1)$. The restriction of the polynomial function t to A_i is injective and $t(A_i) = \mathbb{C} \setminus \{0\}$ for $i = 1, 2$.*

Proof. We claim that $(f,h)^{-1}(B) = f^{-1}(0)$. Indeed, suppose that $f \neq 0$ and $f - h(h+1) = 0$. Substituting $f = h(h+1)$ into (4) gives $ft = 0$. Hence $t = 0$ and by (1) $h = 0$. Therefore $f = h(h+1) = 0$ – contrary to our assumption.

An easy computation shows that the mapping $\mathbb{C}^2 \setminus B \ni (\bar f, \bar h) \to ((\bar h + 1)\bar f/(\bar f - \bar h(\bar h + 1))^2, (\bar f - \bar h^2)(\bar f - \bar h(\bar h + 1))^2/\bar f^2) \in \mathbb{C}^2$ composed with (f,g) is an identity on $\mathbb{C}^2 \setminus B$. This together with the claim gives the first part of the lemma.

Now we describe a curve $f^{-1}(0)$. Assume that $f(x,y) = 0$. Then by (1) $xt + 1 = 0$ or $t^2 + y = 0$. We obtain by (1) in the first case $x = -1/t$, $y = -t(t+1)$, $h = 0$ and in the second case $x = -(t+1)/t^2$, $y = -t^2$, $h = -1$. This finishes the proof. ■

Lemma 3.3 *The complex Pinchuk mapping $\mathbb{C}F$ restricted to $f^{-1}(0)$ is two-to-one between $f^{-1}(0)$ and $\{-1, 0\} \times (\mathbb{C} \setminus \{0\})$.*

Proof. By (3.2) $q(x,y) = -t^2 - 6th(h+1) + u(f,h) = -t^2$ for $(x,y) \in f^{-1}(0)$. Thus by the second part of Lemma 3.2 mappings $(f,h)^{-1}(0,0) \to \{0\} \times (\mathbb{C} \setminus \{0\})$ and $(f,h)^{-1}(0,-1) \to \{-1\} \times (\mathbb{C} \setminus \{0\})$ induced by $\mathbb{C}F$ are two-to-one. ■

Lemma 3.4 *The complex Pinchuk mapping $\mathbb{C}F$ restricted to $\mathbb{C}^2 \setminus f^{-1}(0)$ is a composition*

$$\mathbb{C}^2 \setminus f^{-1}(0) \xrightarrow{(f,h)} \mathbb{C}^2 \setminus B \xrightarrow{G} \mathbb{C}^2$$

where $(a,b) = G(\bar f, \bar h)$ is given by (7).

Proof. It follows immediately from 3.1 and 3.2. ■

Lemma 3.5 *Let $(\bar f, \bar h, a, b)$ satisfy (6) and let $(\bar f, \bar h) \in B$. Then*

(i) if $\bar f = 0$ then $a = -1$ or $a = 0$,

(ii) if $\bar f \neq 0$, $\bar f = \bar h(\bar h + 1)$ then $(a,b) = \Phi(\bar h)$.

Proof. Putting $\bar f = 0$ in (6) we get $\bar h = a$, $Q(0, \bar h, b) = \bar h^4(\bar h + 1)^2$ and (i) follows. To prove (ii) put $\bar f = \bar h(\bar h + 1)$ in (7). We obtain $a = \bar f + \bar h$, $b = u(\bar f, \bar h)$. Thus $(a,b) = \Phi(\bar h)$. ■

Proof of Proposition 2.4. First we show that

$$S_{\mathbb{C}F} \subset (\{-1\} \times \mathbb{C}) \cup (\{0\} \times \mathbb{C}) \cup \Phi(\mathbb{C}).$$

Fix $(a_0, b_0) \in \mathbb{C}^2$ lying in the complement of above curves and take a neighborhood U of (a_0, b_0) such that $cl(U)$ is a compact set disjoint from $(\{-1\} \times \mathbb{C}) \cup (\{0\} \times \mathbb{C}) \cup \Phi(\mathbb{C})$. Let V be the set of all $(\bar{f}, \bar{h}) \in \mathbb{C}^2$ such that $(\bar{f}, \bar{h}, a, b)$ satisfies (6) for some $(a, b) \in cl(U)$. By continuity of roots (see [1], Proposition 1.5.5) V is a compact set. Moreover, it follows from Lemma 3.5 that $V \cap B = \emptyset$. Hence $V = G^{-1}(cl(U))$. Since $\mathbb{C}F^{-1}(cl(U)) = (f, h)^{-1}(V)$, we conclude by Lemma 3.2 that $\mathbb{C}F^{-1}(cl(U))$ is compact. Therefore the map $\mathbb{C}F$ is proper at the point (a_0, b_0).

Now we show that the polynomial $W(\bar{f}, a, b)$ has no multiple factor in a ring $\mathbb{C}[\bar{f}, a, b]$. Assume that it has. Then $Q(\bar{f}, \bar{h}, b) = W(\bar{f}, \bar{f} + \bar{h}, b)$ has also a multiple factor. Since Q can be rewritten as $Q(\bar{f}, \bar{h}, b) = \bar{f}^2 b + Q_1(\bar{f}, \bar{h})$ the only candidate for a multiple factor is $\bar{f}$. But $\bar{f}$ does not divide $Q(\bar{f}, \bar{h}, b)$ contrary to our assumption.

Thus for a generic $(a, b) \in \mathbb{C}^2$ the polynomial $W(\bar{f}, a, b)$ has 6 single roots. It follows that $\#\mathbb{C}F^{-1}(a, b) = 6$ for (a, b) in a general position.

Our next task is to show that lines $\{0\} \times \mathbb{C}$ and $\{-1\} \times \mathbb{C}$ are not contained in $S_{\mathbb{C}F}$. We will use the following property (see [5]): *If* $\#\mathbb{C}F^{-1}(a, b) = \#\mathbb{C}F^{-1}(a_{\text{gen}}, b_{\text{gen}})$, *then* $\mathbb{C}F$ *is proper at a point* (a, b). Here $(a_{\text{gen}}, b_{\text{gen}})$ denotes a point in a general position, so $\#\mathbb{C}F^{-1}(a_{\text{gen}}, b_{\text{gen}}) = 6$.

We have $W(\bar{f}, 0, b) = \bar{f}^2(-197/4\bar{f}^4 + 104\bar{f}^3 - 63\bar{f}^2 + b)$. Thus for a generic $b \in \mathbb{C}$ a system (6) has 4 solutions $(\bar{f}_i, \bar{h}_i)$ $(i = 1, \ldots, 4)$ such that $(\bar{f}_i, \bar{h}_i) \notin B$ and one double solution $(\bar{f}_5, \bar{h}_5) = (0, 0)$. By 3.3 and 3.4 we get $\#\mathbb{C}F^{-1}(0, b) = 6$. Hence by the above mentioned property $\{0\} \times \mathbb{C} \not\subset S_{\mathbb{C}F}$. We show in the same way that $\{-1\} \times \mathbb{C} \not\subset S_{\mathbb{C}F}$. From 2.3 it follows that the set $S_{\mathbb{C}F}$ is either a curve $\Phi(\mathbb{C})$ or is empty. The latter is impossible because in this case real Pinchuk's map would be proper and thus it would be a diffeomorphism. ■

From now we treat all polynomials under considerations as real polynomials.

Lemma 3.6 *The parameterization* Φ *of a curve* C *is injective and* C *is smooth except a point* $(-1, 0)$.

Proof. Suppose to the contrary that there are $s_1 \neq s_2$ such that $\Phi(s_1) = \Phi(s_2)$. An equation $s_1^2 + 2s_1 = s_2^2 + 2s_2$ gives $s_2 = -2 - s_1$. Combining this with equality $u(s_1^2 + s_1, s_1) = u(s_2^2 + s_2, s_2)$ we get $(75s_1^2 + 150s_1 + 104)(s_1 + 1)^3 = 0$. Hence $s_1 = s_2 = -1$ which contradicts the choice of s_1 and s_2.

Since $\frac{d}{ds}\Phi(s)$ vanishes for $s = -1$ only, the curve C has a singularity at most at $(-1, 0)$. ■

Lemma 3.7 *Let* $H : \mathbb{R}^n \to \mathbb{R}^n$ *be a polynomial function which is a local diffeomorphism. Then the function* $\mathbb{R}^n \ni v \to \#H^{-1}(v) \in \mathbb{Z}$ *is lower semi-continuous. Moreover,* H *is proper at a point* $v_0 \in \mathbb{R}^n$ *if and only if* $\#H^{-1}(v)$ *is locally constant at* v_0.

The purely topological proof is left to the reader.

Proof of Theorem 2.1. It follows from 2.2 and 2.4 that S_F is a subset of $\Phi(\mathbb{C}) \cap \mathbb{R}^2$. Since an asymptotic variety does not have isolated points (see [6],

Theorem 3), $S_F = \Phi(\mathbb{R}) = C$. From the topological point of view $C \subset \mathbb{R}^2$ is an embedded line. Hence $\mathbb{R}^2 \setminus C$ has 2 connected components. We shall calculate a multiplicity of fibers in each of them.

The curve C cuts a line $\{3\} \times \mathbb{R}$ transversely at points $(3, 3142) = \Phi(1)$ and $(3, 8406) = \Phi(-3)$. Hence the points $(3, 0)$ and $(3, 4000)$ lie on opposite sides of C. One checks that polynomials $W(\bar{f}, 3, 0)$ and $W(\bar{f}, 3, 4000)$ have 2 real roots (e.g. we can use Sturm sequence). Hence $\#F^{-1}(3, 0) = \#F^{-1}(3, 4000) = 2$ and by Lemma 3.7 $\#F^{-1}(v) = 2$ for every $v \in \mathbb{R}^2 \setminus C$.

Now we show that $F^{-1}(0, 0) = F^{-1}(-1, 0) = \emptyset$. We have $W(\bar{f}, 0, 0) = -\bar{f}^4(197/4\bar{f}^2 - 104\bar{f} + 63\bar{f})$. The only real root of this polynomial is $\bar{f} = 0$. Since by 3.3 $(0, 0)$ does not belong to the set $F(f^{-1}(0))$ we have $F^{-1}(0, 0) = \emptyset$. We check similarly that $F^{-1}(-1, 0) = \emptyset$.

It remains to compute $\#F^{-1}(v)$ for $v \in C$. Let $D(a, b)$ be a discriminant of the polynomial $W(\bar{f}, a, b)$ with respect to $\bar{f}$. Using any computer symbolic algebra program one can check that the polynomial D is nonzero at a point $\Phi(1) \in C$. Hence D does not vanish on C but a finite number of points. Fix $(a_0, b_0) = \Phi(s) \in C$ for which $D(a_0, b_0) \neq 0$. We have shown before that $W(\bar{f}, a, b)$ has two real roots for $(a, b) \in \mathbb{R}^2 \setminus C$. Therefore by continuity of roots the system (6) has two real solutions at a point (a_0, b_0). By 3.5 and 3.6 one of these solutions is $(\bar{f}, \bar{h}) = (s(s + 1), s) \in B$. Thus $\#F^{-1}(a_0, b_0) = \#(G^{-1}(a_0, b_0) \cap \mathbb{R}^2) = 1$.

By Lemma 3.7 we see that $\#F^{-1}(v) \leq 1$ for $v \in C$. Let

$$K = \{v \in C : \#F^{-1}(v) = 0\}.$$

We have shown that K is a finite set and points $(-1, 0)$, $(0, 0)$ belong to K.

The restriction of F from $F^{-1}(C)$ to $C \setminus K$ is a diffeomorphism. Hence the curve $F^{-1}(C)$ has $\#K + 1$ connected components homeomorphic to a line and its complement $\mathbb{R}^2 \setminus F^{-1}(C)$ has $\#K + 2$ connected components.

The set $\mathbb{R}^2 \setminus C$ has two connected components S_1, S_2. Write $F^{-1}(S_i) = \bigcup_{j=1}^{r_i} S_{i,j}$ where $S_{i,j}$ are connected components of the set $F^{-1}(S_i)$. The mappings $S_{i,j} \to S_i$ $(i = 1, 2,\ j = 1, \ldots, r_i)$ induced by F are proper and unramified hence they are topological coverings. Since sets S_i $(i = 1, 2)$ are simply connected, these mappings are homeomorphisms. Since $\#F^{-1}(v) = 2$ for $v \in \mathbb{R}^2 \setminus C$, $F^{-1}(S_i)$ consists of 2 connected components for $i = 1, 2$. It follows that $\#K = 2$, therefore $K = \{(-1, 0), (0, 0)\}$. ∎

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